

PROJECTIVE EMBEDDING OF LOG RIEMANN SURFACES AND K-STABILITY

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ABSTRACT. Given a smooth polarized Riemann surface (X, L) endowed with a hyperbolic metric ω with cusp singularities along a divisor D , we show the L^2 projective embedding of (X, D) defined by L^k is asymptotically almost balanced in a weighted sense. The proof depends on sufficiently precise understanding of the behavior of the Bergman kernel in three regions, with the most crucial one being the neck region around D . This is the first step towards understanding the algebro-geometric stability of extremal Kähler metrics with singularities.

1. INTRODUCTION

Let (X, L) be an n dimensional polarized Kähler manifold. The famous Yau-Tian-Donaldson conjecture relates the existence of constant scalar curvature Kähler (cscK) metrics in the class $2\pi c_1(L)$ to the K-stability of (X, L) . This is essentially a correspondence between differential geometry/PDE and algebraic geometry of (X, L) . The direction from cscK metrics to K-stability was established by Donaldson [12], Stoppa [23], Mabuchi [20], using the idea of *quantization*. The other direction is much more involved, and it has been established for toric surfaces by Donaldson [11], and for anti-canonically polarized Fano manifolds by the recent result of Chen-Donaldson-Sun [5–7] (the corresponding metrics are *Kähler-Einstein*).

A crucial ingredient in the proof of [5–7] is the introduction of a smooth divisor $D \in |-mK_X|$ for some $m \geq 1$. Both aspects of the above conjecture extend naturally to the pair (X, D) with an extra parameter $\beta \in [0, 1]$. On the algebraic geometric side we have a notion of *logarithmic K-stability* for (X, D, K_X^{-1}, β) , and on the differential geometric side the corresponding object is a Kähler-Einstein metric with cone angle $2\pi\beta$ along D . Roughly speaking the strategy of [5–7] is a continuous deformation from $\beta = 0$ to $\beta = 1$. A simple but important fact is that the logarithmic K-stability is linear in β , and it is then evidently important to study both aspects at $\beta = 0$. On the metric side one expects complete Kähler-Einstein metrics on the complement $X \setminus D$, and such metrics are known to exist ([8, 14, 28]), by adapting Yau’s solution of the Calabi conjecture and following Calabi’s ansatz; on the algebraic side the K-semistability of $(X, D, K_X^{-1}, 0)$ is established by [24], [21], [3], [15]. However, a direct relationship between these two facts seems missing.

Now for a general polarized manifold (X, L) with a smooth divisor D the above discussion can be extended in a straightforward way by replacing K_X^{-1} with L . Such a theory has not yet been satisfactorily established. In this direction we expect the following

Conjecture 1.1. *Let (X, L) be a polarized Kähler manifold of dimension n , and D a smooth divisor in the class $c_1(L)$. Denote $\sigma = -(K_X + L) \cdot L^{n-1}/L^n$, and suppose D admits a constant scalar curvature Kähler metric $\omega_D \in 2\pi c_1(L|_D)$. Then $(X, D, L, 0)$ is logarithmic K-semistable if $\sigma \leq 0$.*

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Notice the sign of σ is the same as the sign of the scalar curvature of ω_D . When $\sigma = 0$ Conjecture 1.1 follows from [24] (the proof there is written assuming D is Calabi-Yau, but it is easy to see one only uses the condition that D is scalar flat). When K_X is proportional to L , Conjecture 1.1 holds by the results of [3, 15, 21]. The conjecture can also be intuitively interpreted as a form of “inversion of adjunction” for K-stability, if one assumes the Yau-Tian-Donaldson conjecture holds in dimension $n-1$. It is an interesting question to ask if the algebro-geometric counterpart can be proved directly. From the differential geometric point of view the conjecture also suggests the existence of complete Kähler metrics with negative constant scalar curvature on the complement $X \setminus D$, which is related to the work of H. Auvray [1].

In this paper we will deal with the case $n = 1$, so X is a smooth Riemann surface, and $D = \sum_{i=1}^d p_i$ is an effective divisor of degree d (all p_i 's are distinct). We call such pair (X, D) a *log Riemann surface*. The condition that $\sigma < 0$ is equivalent to that $d > \chi(X)$. Conjecture 1.1 in this case follows from the aforementioned results. However, the proofs in [3, 15, 21] all depend crucially on the special feature that the canonical bundle of X is definite, so seems difficult to be adapted to the general case. Our proof here is based on the quantization technique and reveals the relationship between logarithmic K-stability and the known complete hyperbolic metric on $X \setminus D$. We hope the techniques developed in this paper could help understand the quantization for other types of singular metrics, for examples, those with cone singularities, and lead to the proof that existence of singular cscK/extremal metrics with prescribed asymptotic behavior implies an appropriately extended notion of K-stability. For metrics with cone singularities or Poincaré type singularities along a divisor this has already been speculated in [13, 25].

Before stating our main result, we recall some known facts and fix some notation. Let V be a subvariety of \mathbb{CP}^N and W a subvariety of V . For $\lambda \in [0, 1]$ we define the λ -center of mass of (V, W) to be

$$\mu(V, W, \lambda) = \lambda \int_V \frac{ZZ^*}{|Z|^2} d\mu_{FS} + (1-\lambda) \int_W \frac{ZZ^*}{|Z|^2} d\mu_{FS} - \frac{\lambda \text{Vol}(V) + (1-\lambda) \text{Vol}(W)}{N+1} Id$$

where $[Z] \in \mathbb{CP}^N$ is viewed as a column vector, and the volume is calculated with respect to the induced Fubini-Study metric. Notice μ always takes value in $\text{Lie}(SU(N+1))$; indeed, by general theory μ can be viewed as the moment map for the action of $SU(N+1; \mathbb{C})$ on a certain *Chow variety*. For $B \in \text{Lie}(SU(N+1))$ we write $\|B\|_2 := \sqrt{\text{Tr} BB^*}$.

A pair (V, W) embedded in \mathbb{CP}^N with vanishing λ -center of mass is called a λ -balanced embedding. We say (V, W) is λ -Chow stable if there is an $A \in SL(N+1; \mathbb{C})$ such that $(A.V, A.W)$ is λ -balanced. and we say (V, W) is λ -Chow semistable if the infimum balancing energy

$$E(V, W, \lambda) := \inf_{A \in SL(N+1; \mathbb{C})} \|\mu(A.V, A.W, \lambda)\|_2$$

vanishes. It is well-known that by the Kempf-Ness theorem, these definitions agree with the usual notion of Chow (semi)-stability of log pairs (see for example [16]). When $\lambda = 1$ the subvariety W can be ignored and this reduces to the standard notion of Chow (semi)-stability.

Now going back to our situation of a polarized manifold (X, L) and a smooth divisor D . We say (X, D, L) is λ -almost asymptotically Chow stable if for all k sufficiently large, under the projective embedding of (X, D) induced by sections of $H^0(X, L^k)$ we have $E(V, W, \lambda) = o(k^{-1})$. By [24] if (X, D, L) is λ -almost asymptotically Chow stable then (X, D, L, β) is K-semistable for $\beta = \frac{3\lambda-2}{\lambda}$. We will not

explicitly make use of the notion of (logarithmic) K-(semi)stability in this article, so we will not elaborate on the definition and we refer the readers to [24].

Restricting to our setting of a log Riemann surface (X, D) , with an ample line bundle L of degree l . In this one dimensional case we do not need to assume $L = [D]$. We denote by ω the complete Kähler metric on $X \setminus D$ with constant negative curvature and total volume $2l\pi$. ω can be considered as a closed Kähler current on X whose cohomology class is $2\pi c_1(L)$. So there is a singular metric h on L such that the curvature of h is ω . For k large, we denote by \mathcal{H}_k the subspace of $H^0(X, L^k)$ consisting of holomorphic sections that are L^2 integrable with respect to the norm defined by h and ω . It is easy to see that \mathcal{H}_k agrees with the image of the map $H^0(X, L^k(-D)) \rightarrow H^0(X, L^k)$ given by multiplication by the defining section s_D for D . For k large, we have an embedding $\Phi_k : X \rightarrow \mathbb{P}\mathcal{H}_k^*$. A choice of orthonormal basis of \mathcal{H}_k determines a Hermitian isomorphism of $\mathbb{P}\mathcal{H}_k^*$ with \mathbb{CP}^{N_k} , up to the $U(N_k + 1)$ action, where $N_k + 1 = \dim \mathcal{H}_k$. In particular, the quantity $\|\mu(X, D, \lambda)\|_2$ is independent of the choice of orthonormal basis. The following is our main result

Theorem 1.2. *Given a log Riemann surface (X, D) with $d \geq \chi(X)$, then for any ample line bundle L over X , (X, D, L) is $\frac{2}{3}$ -almost asymptotically Chow stable. More precisely, we have*

$$\|\mu(\Phi_k(X), \Phi_k(D), \frac{2}{3})\|_2^2 = O(k^{-3/2}(\log k)^{121}).$$

We remark here that the exponent 121 suffices for our purpose, but it is far from sharp. It can be improved if necessary.

An immediate corollary, using [24], is that

Corollary 1.1. *$(X, D, L, 0)$ is logarithmic K-semistable.*

Remark: We mention that the first part of Theorem 1.2 and hence corollary 1.1 were also proved in [16], using explicit Hilbert-Mumford criterion and the special feature in complex dimension one. As mentioned above, the main interest in our paper is indeed the second part of Theorem 1.2 on the quantitative estimate of the balancing energy of the L^2 embedding induced by the hyperbolic metric. We hope this will have applications in higher dimensions.

Now we briefly describe the idea of the proof of Theorem 1.2. Let $\{s_\alpha\}$ be an orthonormal basis of \mathcal{H}_k . An important quantity is the “density of state function” (or the *Bergman Kernel function*)

$$\rho_k = \sum_i |s_\alpha|_h^2.$$

Denote $\omega_k = \Phi_k^* \omega_{FS}$ (here our convention is that $\omega_{FS} \in c_1(O(1))$), then

$$2\pi\omega_k = k\omega + i\partial\bar{\partial} \log \rho_k.$$

We know by definition

$$(1.1) \quad \int_X \langle s_\alpha, s_\beta \rangle_h \omega = \delta_{\alpha\beta},$$

and we can write

$$(1.2) \quad \mu(X, D, \lambda) = \lambda \int_X \langle s_\alpha, s_\beta \rangle_h \rho_k^{-1} \omega_k + (1-\lambda) \sum_\alpha \rho_k(p_i)^{-1} \langle s_\alpha(p_i), s_\beta(p_i) \rangle_h - c_k I,$$

where $c_k = \frac{\lambda kl + (1-2\lambda)d}{N_k + 1}$. Since D consists of points, the key is to understand the first term of (1.2). Comparing with (1.1), it is then important to know the behavior

of $\rho_k^{-1}\omega_k$. Not surprisingly, as in the case without divisor, we need to study the function ρ_k .

If ω were a smooth Kähler metric on X , it would follow from the result of Tian, Zelditch, Lu [4, 18, 19, 26, 29], that ρ_k has an asymptotic expansion of the form

$$(1.3) \quad \rho_k = \frac{1}{2\pi} \left[k + \frac{S(\omega)}{2} + O(k^{-1}) \right],$$

Now as observed in [10] this result can be localized. The basic point is that for any $p \in X$ away from D , we have

$$\rho_k(p) = \sup\{|s(p)|^2 | s \in \mathcal{H}_k, \|s\| = 1\}.$$

and the supreme is achieved by a so-called *peak section*. When k is sufficiently large, the rescaled manifold $(X, p, L^k, h^{\otimes k}, k\omega)$ is close to the standard Gaussian model $(\mathbb{C}, 0, L_0, h_0, \omega_0)$, where L_0 is the trivial line bundle over \mathbb{C} , h_0 is the (non-trivial) hermitian metric $e^{-|z|^2/2}$ whose curvature ω_0 is the standard flat metric. This fact allows a construction of the peak section at p by a grafting and perturbation procedure. Everything is local in p except the perturbation involves Hörmander's L^2 estimate which depends on the global lower bound of curvature (this is automatically satisfied in our case). A more careful analysis shows that the expansion (1.3) indeed holds for points p whose injectivity radius is bounded below by $k^{-1/2} \log k$.

The new feature arises when we want to understand ρ_k at the points with small injectivity radius (i.e. points very close to D). A difficult point here is that D has co-dimension one. If D were of higher co-dimension, then by the result of [10] one can ignore a neighborhood of D and obtain a better estimate than that is stated in Theorem 1.2.

Notice ρ_k is unbounded near p_i , and we can not expect the same expansion as (1.3) to hold. Instead we need to look at a different model, which is the punctured hyperbolic disc \mathbb{D}^* . In Section 2 we will analyze the behavior of ρ_k and $\rho_k^{-1}\omega_k$ in the model case. As our investigation shows, there are also two further distinct behavior according to the size of the injectivity radius. For a point p with injectivity radius smaller than $k^{-1/2}(\log k)^{-1}$ we show that the function ρ_k is essentially governed by at most three monomial sections (so we can intuitively think of these are sections “peaked” around a circle instead of at one point); for a point with injectivity radius between $[k^{-1/2}(\log k)^{-1}, k^{-1/2} \log k]$, there are infinitely many monomial sections contributing to ρ_k , and we need to do a much more careful analysis to get the required estimates.

In Section 3 we will use the results of Section 2 to prove Theorem 1.2. In Section 4, we will study for the case $X = \mathbb{P}^1$ and $L = O(d)$, the exact range of $\lambda \in [0, 1]$ for which (X, D) is λ -Chow stable under the embedding induced by L^k . The key point is that for the minimum such λ , which we denote λ_k , we need to construct a degeneration of (X, D) to a λ_k -balanced pair (X_0, D_0) . We will show that $\lambda_k < 2/3$ and prove the existence of such pair. It turns out that the degeneration is exactly given by deformation to the normal cone of D , so X_0 consists of $d+1$ components, one isomorphic to X , and the other are d lines. This contrasts the case considered in [24] (see also Figure 1 and Figure 2), when $\sigma = 0$ (in the one dimensional case, this means $X = \mathbb{P}^1$ and D consists of two points). In that case the limiting balanced pair consists of a chain X_0 of k lines in \mathbb{P}^k , so the number of components goes to infinity as k tends to infinity. One would expect the same picture to also hold in higher dimension. This difference should also reflect the interesting facts that the complete negative Kähler-Einstein metrics constructed in [8, 14, 28] has finite volume, while the Tian-Yau complete Ricci-flat metric constructed in [27] has infinite volume.

The draft of this paper was finished around November 2015. Recently we are informed of the paper by Auvray-Ma-Marinescu [2], which studies the Bergman kernels on punctured Riemann surfaces. There are also many results in the literature studying the asymptotics of Bergman kernels of singular Kähler metrics, see for example [9, 17, 22]. Our paper has different motivation from these and for our geometric purpose we need more refined information of the Bergman kernel.

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2. CALCULATION FOR THE MODEL

We will denote by $\varepsilon(k)$ a quantity depending on k that is $O(k^{-m})$ as $k \rightarrow \infty$, for all $m \geq 0$. Recall that our model is the punctured disk $\mathbb{D}^* = \{z \in \mathbb{C} \mid |z| \leq 1\}$, endowed with the Kähler metric

$$(2.1) \quad \omega_0 = \frac{idz \wedge d\bar{z}}{|z|^2 (\log \frac{1}{|z|^2})^2}.$$

The corresponding Kähler potential is $\Phi_0 = -\log \log \frac{1}{|z|^2}$, and the scalar curvature of ω_0 is -2 . For $k \geq 1$, we let $\mathcal{H}_{k,0}$ be the Bergman space of holomorphic functions f on \mathbb{D}^* such that

$$\|f\|_k^2 := \int_{\mathbb{D}^*} |f|^2 e^{-k\Phi_0} \omega_0 < \infty.$$

On $\mathcal{H}_{k,0}$ we denote by $\langle \cdot, \cdot \rangle_k$ the corresponding Hermitian inner product.

Lemma 2.1. *For any $a \geq 1$, we have $z^a \in \mathcal{H}_{k,0}$ and*

$$(2.2) \quad \langle z^a, z^b \rangle_k = \frac{2\pi(k-2)!}{a^{k-1}} \delta_{ab}.$$

In particular, the functions $\{(\frac{a^{k-1}}{2\pi(k-2)!})^{1/2} z^a \mid a \geq 1\}$ form an orthonormal basis of $\mathcal{H}_{k,0}$

Proof. First of all, it is easy to see that z^a is L^2 with respect to the given weight if and only if $a \geq 1$. By the S^1 symmetry of the metric and weight, z^a 's are obviously orthogonal to each other. Now we calculate the norms:

$$\begin{aligned} \|z^a\|_k^2 &= \int_{\mathbb{D}^*} |z|^{2a} (\log \frac{1}{|z|^2})^k \frac{idz d\bar{z}}{|z|^2 (\log \frac{1}{|z|^2})^2} = \int_{\mathbb{D}^*} |z|^{2(a-1)} (\log \frac{1}{|z|^2})^{k-2} idz d\bar{z} \\ &= 2\pi \int_0^1 x^{a-1} (-\log x)^{k-2} dx \end{aligned}$$

Using the substitution $t = -\log x$, we get

$$\|z^a\|_k^2 = 2\pi \int_0^\infty t^{k-2} e^{-at} dt = \frac{2\pi(k-2)!}{a^{k-1}}$$

□

Remark: The calculation above actually shows us more. By the substitution $t = (k-2)y$, we get that

$$\int_0^\infty t^{k-2} e^{-at} dt = (k-2)^{k-1} \int_0^\infty e^{(k-2)(\log y - ay)} dy.$$

So Laplace's method tells that for large k the integral is concentrated in a small neighborhood of $t = \frac{k-2}{a}$, i.e. $|z|^2 = e^{-(k-2)/a}$. Moreover, the concentration is within a neighborhood of radius $\frac{k^{1/2} \log k}{a}$ of $t = \frac{k-2}{a}$, i.e.

$$(2.3) \quad \int_{|t - \frac{k-2}{a}| \leq \frac{k^{1/2} \log k}{a}} t^{k-2} e^{-at} dt \geq (1 - \varepsilon(k)) \int_0^\infty t^{k-2} e^{-at} dt.$$

From the lemma we see the Bergman kernel of $\mathcal{H}_{k,0}$ is given by

$$(2.4) \quad \rho_{k,0} = \frac{(\log 1/|z|^2)^k}{2\pi(k-2)!} \sum_{a=1}^\infty a^{k-1} |z|^{2a}.$$

By the preceding remark, we see that near the origin, only those terms of small degrees matter. So we can heuristically view $\rho_{k,0}$ as a polynomial function in $|z|^2$. Formally the above orthonormal basis of $\mathcal{H}_{k,0}$ induces an embedding of \mathbb{D}^* into an infinite dimensional complex projective space, and the induced metric is given by

$$(2.5) \quad \omega_{k,0} := \frac{1}{2\pi} (k\omega_0 + i\partial\bar{\partial} \log \rho_{k,0}) = \frac{1}{2\pi} i\partial\bar{\partial} \log \sum_{a=1}^\infty a^{k-1} |z|^{2(a-1)}.$$

Our main goal in this section is to understand $\rho_{k,0}$ and $\omega_{k,0}$. This serves as a local model for understanding the Bergman kernel and the induced Fubini-Study metric near a hyperbolic cusp in our setup described in the introduction.

To simplify notation we will denote $x = |z|^2$, and we will shift k by 1 (so we are studying instead $\mathcal{H}_{k+1,0}$). We write

$$\varphi_k(x) = \sum_{a=1}^\infty a^k x^{a-1},$$

then $\omega_{k+1}(x) = \frac{1}{2\pi} \varphi_k^{-2} \psi_k i dz d\bar{z}$, where

$$\psi_k = \varphi_k \Delta_z \varphi_k - |\partial_z \varphi_k|^2$$

The integral in the model case corresponding to the one we are interested in (1.2) is the following

$$(2.6) \quad \mu_a := \int_{\mathbb{D}^*} \left(\frac{|z^a|}{\|z^a\|_{k+1}} \right)^2 e^{-(k+1)\Phi_0} \rho_{k+1,0}^{-1} \omega_{k+1} = \int_0^1 x^{a-1} a^k \varphi_k(x)^{-3} \psi_k(x) dx.$$

Similarly to the compact case, to measure the deviation of the image of \mathbb{D}^* in the infinite dimensional projective space from being $\frac{2}{3}$ -balanced, we need to estimate $\frac{2}{3}\mu_a + \frac{1}{3}\delta_{a1}$. We divide into three cases

Case I: $a \geq k^{1/2} \log k$. In this case by the above remark z^a is concentrated at points z with $|z|^2$ approximately $e^{-k^{1/2}(\log k)^{-1}}$. The injectivity radius of the metric ω_0 at these points is approximately $\pi(\log \frac{1}{|z|^2})^{-1} \approx \pi k^{-1/2} \log k$. Then as mentioned in the introduction, when $|z|^2 \geq e^{-k^{1/2}(\log k)^{-1}}$, the usual proof of the Bergman kernel expansion (c.f. [10]) goes through, and provide a uniform estimate

$$\rho_{k,0} = \frac{1}{2\pi} (k-1 + O(k^{-1})),$$

which holds in the C^2 sense. This implies

$$\omega_{k,0} = \omega_0(1 + O(k^{-2})),$$

From these we obtain that $\mu_a = 1 + O(k^{-2})$ for $a \geq k^{1/2} \log k$. Moreover, this argument also gives rise the following estimate of the volume of $\omega_{k,0}$.

Lemma 2.2.

$$(2.7) \quad \int_{|z|^2 \leq e^{-k^{1/2}(\log k)^{-1}}} \omega_{k,0} = O(k^{-1/2} \log k).$$

Proof. By definition $\omega_{k,0} = \frac{1}{2\pi} k \omega_0 + \frac{1}{2\pi} i \partial \bar{\partial} \log \rho_{k,0}$. By direct calculation,

$$\int_{|z|^2 \leq e^{-k^{1/2}(\log k)^{-1}}} \omega_0 = k^{-1/2} \log k.$$

For the other term, using integration by parts and the above expansion of $\rho_{k,0}$, we have

$$\left| \int_{|z|^2 \leq e^{-k^{1/2}(\log k)^{-1}}} i \partial \bar{\partial} \log \rho_{k,0} \right| \leq \left| \int_{|z|^2 \leq e^{-k^{1/2}(\log k)^{-1}}} J d \rho_{k,0} \right| = O(k^{-2} \log k).$$

□

Case II: $a = o(k^{1/2}(\log k)^{-1/2})$. In this case the sections z^a are concentrated in a very small neighborhood of 0. By a direct calculation we have

$$\begin{aligned} \psi_k(x) &= \sum_{a=1, b=2} a^k b^k (b-1)^2 x^{a+b-3} - \sum_{a=2, b=2} a^k b^k (a-1)(b-1) x^{a+b-3} \\ &= \sum_{b=2} b^k (b-1)^2 x^{b-2} + \sum_{a=2, b=2} a^k b^k (b-1)(b-a) x^{a+b-3} \\ &= \sum_{l=3} c_l x^{l-3}, \end{aligned}$$

where

$$\begin{aligned} c_l &= \sum_{a+b=l, a \geq 2, b \geq 2} a^k b^k (b-1)(b-a) + (l-2)^2 (l-1)^k \\ &= \sum_{a+b=l, a \geq 1, b \geq 2} a^k b^k (b-1)(b-a) \\ &= \sum_{a+b=l, a \geq 1, a < b} a^k b^k (b-a)^2 \end{aligned}$$

As power series, φ_k and ψ_k are complicated to use in calculating integrals. Our first observation is that $\varphi_k(x)$ can be estimated using only 2 or 3 terms when x is small. More precisely:

Lemma 2.3. *When $x \in [0, 2^{-k}]$, $\varphi_k(x) = 1 + 2^k x + \varepsilon(k)$. Similarly, for $x \in [\frac{n+1}{n}]^{-k}, (\frac{n+2}{n+1})^{-k}]$, we have*

$$\varphi_k(x) = (1 + \varepsilon(k))(n^k x^{n-1} + (n+1)^k x^n + (n+2)^k x^{n+1})$$

as long as $n^2 = o(\frac{k}{\log k})$.

Proof. The quotient of two adjacent terms is $\frac{(a+1)^k x}{a^k}$. Suppose $x \leq 2^{-k}$. Notice $\frac{a+1}{2a} < \frac{3}{4}$ for $a \geq 3$. So

$$\sum_{a \geq 3} a^k x^{a-1} \leq \left(\frac{3}{4}\right)^k \sum_{a \geq 0} \left(\frac{3}{4}\right)^a = \varepsilon(k).$$

Now suppose $(\frac{n+1}{n})^{-k} \leq x \leq (\frac{n+2}{n+1})^{-k}$ for some integer n . For $a \geq n+2$, we have

$$\frac{(a+1)(n+1)}{a(n+2)} \leq 1 - \frac{1}{(n+2)^2}.$$

So as long as $n^2 = o(\frac{k}{\log k})$, $\frac{(a+1)^k x}{a^k} = \varepsilon(k)$. Then

$$\sum_{a \geq n+3} a^k x^{a-1} = (1 + \varepsilon(k))(n+2)^k x^{n+1}.$$

Similarly, for $a \leq n-1$, we have

$$\sum_{a \leq n-1} a^k x^{a-1} = (1 + \varepsilon(k))n^k x^{n-1}.$$

The lemma is then proved. \square

Now we consider $\psi_k(x)$. As in the proof of the preceding lemma, we first notice that c_l is dominated by the middle terms as long as $l^2 = o(\frac{k}{\log k})$. More precisely, when l is odd,

$$c_l = (1 + \varepsilon(k))(\lfloor \frac{l}{2} \rfloor \lceil \frac{l}{2} \rceil)^k$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ means round-down and round-up respectively. When l is even,

$$c_l = (1 + \varepsilon(k))4 \cdot ((l/2 - 1)(l/2 + 1))^k.$$

With these in mind, we can now approximate $\psi_k(x)$. More precisely, we have

Lemma 2.4. *When $x \leq (\sqrt{3})^{-k}$, $\psi_k(x) = (1 + \varepsilon(k))(2^k + 6^k x^2)$. Similarly, for $x \in [(\sqrt{\frac{n+2}{n}})^{-k}, (\sqrt{\frac{n+3}{n+1}})^{-k}]$, we have*

$$\psi_k(x) = (1 + \varepsilon(k))((n(n+1))^k x^{2(n-1)} + ((n+1)(n+2))^k x^{2n} + ((n+2)(n+3))^k x^{2(n+1)})$$

as long as $n^2 = o(\frac{k}{\log k})$.

Proof. The proof is similar to that for $\varphi_k(x)$. We only want to remind the reader that the odd power terms are omitted. The reason is that within each interval appeared in the lemma the odd power terms are dominated by the adjacent even power terms. \square

Lemma 2.3 describes a set of ladders a_n^{-k} for $\varphi_k(x)$, where $a_n = \frac{n+1}{n}$, $n \geq 1$. Lemma 2.4 describes a set of ladders b_n^{-k} for $\psi_k(x)$, where $b_n = \sqrt{\frac{n+2}{n}}$, $n \geq 1$. It is immediate to see that we have

$$a_n > b_n > a_{n+1}$$

Since the integral we are interested in involves both $\varphi_k(x)$ and $\psi_k(x)$ and we want to use the approximations given by lemma 2.3 and lemma 2.4, we will further refine our intervals to that of the form (a_n^{-k}, b_n^{-k}) and (b_n^{-k}, a_{n+1}^{-k}) . The following is a direct consequence of Lemma 2.3 and Lemma 2.4.

Lemma 2.5. *• Within the interval $[a_n^{-k}, b_n^{-k}]$, we have*

$$\begin{aligned} \varphi_k(x) &= (1 + \varepsilon(k))(n^k x^{n-1} + (n+1)^k x^n) \\ \psi_k(x) &= (1 + \varepsilon(k))((n(n+1))^k x^{2(n-1)} + ((n+1)(n+2))^k x^{2n}) \end{aligned}$$

• Within the interval $[b_n^{-k}, a_{n+1}^{-k}]$, we have

$$\begin{aligned} \varphi_k(x) &= (1 + \varepsilon(k))((n+1)^k x^n + (n+2)^k x^{n+1}) \\ \psi_k(x) &= (1 + \varepsilon(k))((n(n+1))^k x^{2(n-1)} + ((n+1)(n+2))^k x^{2n}) \end{aligned}$$

Now we are ready to evaluate integrals.

Proposition 2.6. i) When $a = 1$, we have

$$\int_0^{2^{-k}} \frac{\psi_k(x)dx}{(\varphi_k(x))^3} = \frac{3}{8} + \varepsilon(k)$$

$$\int_{2^{-k}}^{(\sqrt{3})^{-k}} \frac{\psi_k(x)dx}{(\varphi_k(x))^3} = \frac{1}{8} + \varepsilon(k)$$

ii) When $a = 2$, we have

$$\int_0^{2^{-k}} 2^k x \frac{\psi_k(x)dx}{(\varphi_k(x))^3} = \frac{1}{8} + \varepsilon(k)$$

$$\int_{2^{-k}}^{(\sqrt{3})^{-k}} 2^k x \frac{\psi_k(x)dx}{(\varphi_k(x))^3} = \frac{3}{8} + \varepsilon(k)$$

$$\int_{(\sqrt{3})^{-k}}^{(3/2)^{-k}} 2^k x \frac{\psi_k(x)dx}{(\varphi_k(x))^3} = \frac{3}{8} + \varepsilon(k)$$

$$\int_{(3/2)^{-k}}^{(\sqrt{2})^{-k}} 2^k x \frac{\psi_k(x)dx}{(\varphi_k(x))^3} = \frac{1}{8} + \varepsilon(k)$$

Proof. For each integral, we replace $\varphi_k(x)$ and $\psi_k(x)$ with the corresponding approximations listed in lemma 2.5. Then by simple substitutions, we can evaluate the integrals. \square

The picture we see for $a = 2$ actually reflects the picture for general a . We will use the following notations:

$$I_{a,n} = \int_{a_n^{-k}}^{b_n^{-k}} (a+1)^k x^a \frac{\psi_k(x)dx}{(\varphi_k(x))^3}$$

$$I'_{a,n} = \int_{b_n^{-k}}^{a_{n+1}^{-k}} (a+1)^k x^a \frac{\psi_k(x)dx}{(\varphi_k(x))^3}$$

Proposition 2.7. For $n \geq 2$

$$I_{n-1,n} = \frac{1}{8} + \varepsilon(k)$$

$$I_{n,n} = \frac{3}{8} + \varepsilon(k)$$

$$I'_{n,n} = \frac{3}{8} + \varepsilon(k)$$

$$I'_{n+1,n} = \frac{1}{8} + \varepsilon(k)$$

Proof. Plugging in the approximations for $\varphi_k(x)$ and $\psi_k(x)$, we get

$$I_{a,n} = \int_{a_n^{-k}}^{b_n^{-k}} \left(\frac{(a+1)(n+1)}{n^2} \right)^k x^{a-n+1} \frac{1 + \left(\frac{n+2}{n+1} \right)^k x^2}{\left(1 + \left(\frac{n+1}{n} \right)^k x \right)^3} dx$$

and

$$I'_{a,n} = \int_{b_n^{-k}}^{a_{n+1}^{-k}} \left(\frac{(a+1)n}{(n+1)^2} \right)^k x^{a-n-2} \frac{1 + \left(\frac{n+2}{n+1} \right)^k x^2}{\left(1 + \left(\frac{n+2}{n+1} \right)^k x \right)^3} dx$$

When $a = n$, we use the substitution $y = (\frac{n+1}{n})^k x$, and get

$$\begin{aligned}
I_{n,n} &= \int_1^d \frac{x(1+bx^2)}{(1+x)^3} dx \\
&= \frac{1+b}{2(1+x)^2} - \frac{1+3b}{1+x} + b(1+x) - 3b \log(1+x) \Big|_1^d \\
&= \left(\frac{1}{2(1+x)^2} - \frac{1}{1+x} \right) \Big|_1^d + \varepsilon(k) \\
&= \frac{3}{8} + \varepsilon(k),
\end{aligned}$$

where $b = (\frac{n+2}{n+1})^k$ and $d = \sqrt{\frac{1}{b}}$.

We can compute the other 3 integrals in the same way, using the fact the integrands are all rational functions. \square

In order to calculate μ_a , we need also calculate the integrals on other intervals. The following lemma tells us that we already have the main value.

Proposition 2.8. *For $n \geq 2$*

$$\begin{aligned}
I_{n-2,n} &= \varepsilon(k) \\
I_{n+1,n} &= \varepsilon(k) \\
I'_{n-1,n} &= \varepsilon(k) \\
I'_{n+2,n} &= \varepsilon(k)
\end{aligned}$$

The calculations are basically the same as that in the last proposition. This proposition shows that the integrals on the nearby intervals are negligible. As we have remarked, the mass of the integrands decay rapidly away from the main intervals. More precisely, we can write

$$\mu_n = \int_0^1 \frac{n^k x^{n-1}}{\varphi_k(x)} \omega_{k,0}.$$

We claim the contribution of the integral from $x \leq a_{n-1}^{-k}$ or $x \geq a_{n+1}^{-k}$ are both $\varepsilon(k)$. Since $a = o(k^{1/2}(\log k)^{-1})$, we know from the definition of $\varphi_k(x)$ that the integrand itself is $\varepsilon(k)$ for x in this region. Now by Lemma 2.2, it follows that the contribution from $(0, a_{n-1}^{-k})$ and $(a_{n+1}^{-k}, e^{-k^{1/2}(\log k)^{-1}})$ to μ_n is $\varepsilon(k)$. When $x > e^{-k^{1/2}(\log k)^{-1}}$, we know from Case I that $\omega_{k,0} \leq 2\rho_{k,0}\omega_0$. But since

$$\int \frac{n^k x^{n-1}}{\varphi_k(x)} \rho_{k,0} \omega_0 = 1,$$

and by (2.3) we know the contribution to this integral from $x > e^{-k^{1/2}(\log k)^{-1}}$ is $\varepsilon(k)$, so the contribution to μ_n is also $\varepsilon(k)$. This proves the claim.

Therefore we obtain

Theorem 2.9. *For $a > 1$ satisfying $a^2 = o(\frac{k}{\log k})$, we have $\mu_a = 1 + \varepsilon(k)$. When $a = 1$, we have $\mu_1 = \frac{1}{2} + \varepsilon(k)$*

Case III: $a \in [k^{1/2}(\log k)^{-1}, k^{1/2} \log k]$. In this case the sections z^a are concentrated in the “neck region”. In order to estimate μ_a , we will compare it with the above standard integral. We may write

$$\gamma_a(x) = \frac{\varphi_k(x)}{a^k x^{a-1}},$$

then

$$\mu_a = \int_0^1 \frac{\Delta_z \log \gamma_a(x) dx}{\gamma_a(x)}.$$

Next we use substitution $v = \log \frac{1}{x}$. Then $\gamma_a(x) = \sum_{c=-a+1}^{\infty} (\frac{a+c}{a})^k e^{-cv}$. Let $v = u + \frac{k}{a}$, we can write

$$\gamma_a(x) = f_a(u) := \sum_{c \geq -a+1} e^{k(\log(1+\frac{c}{a}) - \frac{c}{a})} e^{-cu},$$

and since $\Delta_z = \frac{1}{x} \frac{d^2}{du^2}$ and $dx = -x du$. We get

$$(2.8) \quad \mu_a = \int_{-k/a}^{\infty} \frac{(\log f_a(u))'' du}{f_a(u)}$$

Lemma 2.10.

$$(2.9) \quad \mu_a = \int_{-(\log k)^2}^{(\log k)^2} \frac{(\log f_a(u))'' du}{f_a(u)} + \varepsilon(k)$$

Proof. The idea again is to view $\frac{1}{f_a(u)}$ as the integrand, with the measure part given by $\omega_{k,0}$. We refer the reader to the arguments right after Proposition 2.8, which also works here.

Since $f_a(u)$ is a convex function in u , and $f_a(0) \geq 1$, we only need to show that when $|u| = (\log k)^2$, we have $\frac{1}{f_a(u)} = \varepsilon(k)$. But this is already clear when we look at the terms when $|c| = 1$. \square

Now we have

$$\int_{-(\log k)^2}^{(\log k)^2} \frac{(\log f_a(u))'' du}{f_a(u)} = \int_{-(\log k)^2}^{(\log k)^2} \frac{f_a''(u) du}{(f_a(u))^2} - \int_{-(\log k)^2}^{(\log k)^2} \frac{(f_a'(u))^2 du}{(f_a(u))^3}$$

The following estimates are based on the simple fact that the function $\log(1+x) - x$ is concave with only a unique maximum at $x = 0$.

Lemma 2.11.

$$\int_{-(\log k)^2}^{(\log k)^2} \frac{f_a''(u) du}{(f_a(u))^2} = 2 \int_{-(\log k)^2}^{(\log k)^2} \frac{(f_a'(u))^2 du}{(f_a(u))^3} + \varepsilon(k).$$

Proof. This is basically integration by parts, since

$$\int_{-(\log k)^2}^{(\log k)^2} \frac{f_a''(u) du}{(f_a(u))^2} = \frac{f_a'(u)}{(f_a(u))^2} \Big|_{-(\log k)^2}^{(\log k)^2} + 2 \int_{-(\log k)^2}^{(\log k)^2} \frac{(f_a'(u))^2 du}{(f_a(u))^3}.$$

So we need to evaluate the boundary values. When $u = (\log k)^2$, it is easy to see that both $f_a(u)$ and $f_a'(u)$ are dominated by the terms with $c \leq 0$. So $|\frac{f_a'(u)}{(f_a(u))^2}| \leq \frac{a}{f_a(u)} = \varepsilon(k)$. Now we consider $u = -(\log k)^2$. Let $P(c) = k(\log(1+\frac{c}{a}) - \frac{c}{a}) - cu$. Then $P'(c) = \frac{k}{a+c} - \frac{k}{a} - u$ and $g''(c) = -\frac{k}{(a+c)^2} < 0$. So $P(c)$ is a concave function of c . When $u = -(\log k)^2$, $f_a(u)$ and $f_a'(u)$ are dominated by the terms with $c > 0$. The zero of $P'(c)$ is $c_0 = \frac{k}{k/a+u} - a = \frac{-au}{\frac{k}{a}+1}$. So $f_a(u)$ is dominated by a term around $c_0 = O((\log k)^4)$ (since c_0 may not be an integer). Thus at $u = -(\log k)^2$, $|\frac{f_a'(u)}{(f_a(u))^2}| \leq \frac{|c_0|}{f_a(u)} = O(\frac{(\log k)^4}{f_a(u)}) = \varepsilon(k)$. And the lemma is proved. \square

From these we obtain

$$\mu_a = \frac{1}{2} \int_{-(\log k)^2}^{(\log k)^2} \frac{f_a''(u) du}{(f_a(u))^2} + \varepsilon(k).$$

We now further simplify the integral. Let

$$g_a(u) = \sum_{|c| \leq (\log k)^5} e^{k(\log(1+\frac{c}{a})-\frac{c}{a})-cu}.$$

Then by similar arguments as in the proof of the previous lemma we see that

$$f_a(u) = g_a(u)(1 + \varepsilon(k)),$$

and

$$f_a''(u) = g_a''(u)(1 + \varepsilon(k)).$$

So

$$(2.10) \quad \mu_a = \frac{1}{2} \int_{-(\log k)^2}^{(\log k)^2} \frac{g_a''(u) du}{(g_a(u))^2} + \varepsilon(k).$$

Now we define

$$h_a(u) = \sum_{c \in \mathbb{Z}} e^{-\frac{kc^2}{2a^2}} e^{-cu}.$$

As before for $u \in [-(\log k)^2, (\log k)^2]$ we have

$$h_a(u) = (1 + \varepsilon(k)) \sum_{|c| \leq (\log k)^5} e^{-\frac{kc^2}{2a^2}} e^{-cu}.$$

Lemma 2.12. *We have*

$$\int_{-(\log k)^2}^{(\log k)^2} \frac{g_a''(u) du}{(g_a(u))^2} = \int_{-\infty}^{(\infty)^2} \frac{h_a''(u) du}{(h_a(u))^2} + O\left(\frac{(\log k)^{60}}{k}\right).$$

Proof. For $u \in [-(\log k)^2, (\log k)^2]$, we have

$$\begin{aligned} g_a(u) &= \sum_{|c| \leq (\log k)^5} e^{-\frac{kc^2}{2a^2}-cu} \left(1 + \frac{kc^3}{a^3} + O\left(\frac{k^2 c^6}{a^6} + \frac{kc^4}{a^4}\right)\right) \\ &= h_a(u) \left(1 + O\left(\frac{(\log k)^{36}}{k}\right)\right) + G_a(u), \end{aligned}$$

where

$$G_a(u) = \sum_{|c| \leq (\log k)^5} e^{-\frac{kc^2}{2a^2}-cu} \frac{kc^3}{a^3} = h_a(u) O\left(\frac{(\log k)^{20}}{k^{1/2}}\right).$$

Similarly

$$g_a''(u) = h_a''(u) \left(1 + O\left(\frac{(\log k)^{46}}{k}\right)\right) + G_a''(u),$$

and

$$G_a''(u) = h_a''(u) O\left(\frac{(\log k)^{30}}{k^{1/2}}\right).$$

Notice both $G_a(u)$ and $G_a''(u)$ are odd functions, so

$$\int_{-(\log k)^2}^{(\log k)^2} \frac{g_a''(u) du}{g_a(u)} = \left(1 + O\left(\frac{(\log k)^{60}}{k}\right)\right) \int_{-(\log k)^2}^{(\log k)^2} \frac{h_a''(u) du}{h_a(u)}.$$

The lemma then follows from the fact that the last integral can be replaced by the integral over $(-\infty, \infty)$, with a possibly $\varepsilon(k)$ error. The proof is similar to the previous arguments, and we omit it here. \square

Now the following elementary lemma is crucial for our purpose.

Lemma 2.13. *For all $a > 0$, we have*

$$\int_{-\infty}^{\infty} \frac{h_a''(u) du}{(h_a(u))^2} = 2.$$

Proof. For simplicity let $b = \frac{k}{2a^2}$, and by abuse of notation we will $h_b(u) = \sum_{c \in \mathbb{Z}} e^{-bc^2} e^{-cu}$. Notice $h_b(x) = H_b(x) e^{-u^2/(4b)}$, where $H_b(x) = \sum_{c \in \mathbb{Z}} e^{-b(c - \frac{u}{2b})^2}$. Since the summation is for all integers, we see that $H_b(u)$ is periodic with period $2b$. So

$$\int_{-\infty}^{\infty} \frac{dx}{h_b(u)} = \int_0^{2b} \frac{\sum_{c \in \mathbb{Z}} e^{-\frac{(u+2bc)^2}{4b}}}{H_b(u)} du = \int_0^{2b} du = 2b.$$

It is easy to justify that differentiating with respect to b commute with the integral, and notice that $h_b(u)$ satisfies the heat equation $\frac{d}{db} h_b(u) = -h_b''(u)$, we obtain

$$\int_{-\infty}^{\infty} \frac{h_b''(u) du}{h_b(u)^2} = 2.$$

□

To sum up, the above discussion yields

Theorem 2.14. *For $a \in [k^{1/2}(\log k)^{-1}, k^{1/2} \log k]$, we have $\mu_a = 1 + O(k^{-1}(\log k)^{60})$.*

From the proof it is easy to see that the same estimate holds if $a \in [C^{-1}k^{1/2}(\log k)^{-1}, Ck^{1/2} \log k]$ for a fixed $C > 0$.

The above discussion of Case III suggests that the behavior of the Bergman kernel on the neck is modeled on a renormalized measure on the infinite cylinder \mathbb{C}^* . Indeed, for any $a \in [k^{1/2}(\log k)^{-1}, k^{1/2} \log k]$ we know the section z^a is concentrated in an annuli neighborhood of the circle $\log \frac{1}{|z|^2} = \frac{k-2}{a}$. If we change to cylindrical coordinates $z = e^{-(\xi/2 + (k-2)/a)}$, where $\xi = u + it$. Then we see the measure $|z|^{2a} e^{-k\Phi_0} \omega_0 = |z|^{2a-2} (\log \frac{1}{|z|^2})^{k-2} dz d\bar{z}$ is to the leading order term approximated by the measure $d\mu_0 = e^{-\frac{a^2 u^2}{2k}} du dt$ on the cylinder. On \mathbb{C}^* , we can define a L^2 norm on the space of all holomorphic functions using the measure $d\mu_0$. It is easy to see

$$\|z^c\|^2 = e^{\frac{kc^2}{2a^2}}.$$

The corresponding Bergman kernel

$$\rho(\xi) = \sum_{c \in \mathbb{Z}} e^{-\frac{a^2 u^2}{2k} - \frac{kc^2}{2a^2} + cu} = \sum_{c \in \mathbb{Z}} e^{-\frac{a^2}{2k} (u - \frac{kc}{a^2})^2}.$$

Our discussion above makes precise that this model approximates $\rho_{0,k}$ when k is large.

3. GENERAL CASE

We use the setup of the introduction. Let (X, D) be a log Riemann surface, and L be an ample line bundle over X . Let $\Phi_k : X \rightarrow \mathbb{CP}^{N_k}$ be the map defined in the introduction. Our goal is to estimate the asymptotics of $\|\mu(\Phi_k(X), \Phi_k(D), \frac{2}{3})\|_2$ as $k \rightarrow \infty$, using a particular choice of orthonormal basis of \mathcal{H}_k .

First given any orthonormal basis $\{s_\alpha\}$ of \mathcal{H}_k , we re-write (1.2) as

$$\frac{3}{2} \mu(X, D, \frac{2}{3}) = \mu_X + \frac{1}{2} \mu_D - \tilde{c}_k I,$$

where

$$\begin{aligned} \mu_X &= \int_X \langle s_\alpha, s_\beta \rangle_h \rho_k^{-1} \omega_k; \\ \mu_D &= \sum_{i=1}^d \rho_k(p_i)^{-1} \langle s_\alpha(p_i), s_\beta(p_i) \rangle_h. \end{aligned}$$

Using Riemann-Roch formula, we obtain

$$\tilde{c}_k = \frac{kl - d + \frac{d}{2}}{kl - d - g + 1} = 1 - \frac{S}{2}k^{-1} + O(k^{-2}),$$

where $S = -\frac{d+2g-2}{l}$ is the scalar curvature of ω , by our normalization.

Now let $D = \{p_1, \dots, p_d\}$. For each i we can find a local holomorphic coordinate chart (U_i, z_i) of X centered at p_i , such that $\omega = -\frac{2}{5}\omega_0$ on U_i , and a local holomorphic section e_i of L over U_i , with $|e_i|^2 = e^{\frac{2}{5}\Phi_0}$, where ω_0 and Φ_0 are defined in the beginning of Section 2. We may assume $U_i = \{|z_i| < 1\}$ and $U_i \cap U_j = \emptyset$ if $i \neq j$. Inside each U_i we are essentially reduced to the model case studied in Section 2, with a possible change of k by $-\frac{2}{5}k$ (notice in the whole discussion there k does not have to be an integer).

Fix a smooth cut-off function χ_i that equals 1 in U_i , and vanishes outside a small neighborhood of U_i . To obtain a global section of L^k , we use Hörmander's L^2 estimate. The following lemma is well-known, see for example [26].

Lemma 3.1. *Suppose that (M, g) is a complete Kähler manifold of complex dimension n , L is a line bundle on M with the hermitian metric h . If*

$$\langle -2\pi i\Theta_h + \text{Ric}(g), v \wedge \bar{v} \rangle_g \geq C|v|_g^2$$

for any tangent vector v of type $(1, 0)$ at any point of M , where $C > 0$ is a constant. Then for any C^∞ L -valued $(0, 1)$ -form α on M with $\bar{\partial}\alpha = 0$ and $\int_M |\alpha|^2 dV_g$ finite, there exists a C^∞ L -valued function β on M such that $\bar{\partial}\beta = \alpha$ and

$$\int_M |\beta|^2 dV_g \leq \frac{1}{C} |\alpha|^2 dV_g$$

where dV_g is the volume form of g and the norms induced by h and g .

Fix k large so that the assumption of the lemma is satisfied for $(X \setminus D, L^k)$. For positive integer $a \leq k^{3/4}$, we apply the lemma to $\alpha_{i,a} = \bar{\partial}(\chi\tau_a z_i^a e_i^{\otimes k})$ (where τ_a is the normalization constant appearing in Lemma 2.1) on $X \setminus D$ and obtain the corresponding $\beta_{i,a}$. Then the section $s_{i,a} := \chi\tau_a z_i^a e_i^{\otimes k} - \beta_{i,a}$ is holomorphic over $X \setminus D$, and the L^2 integrability condition guarantees that $s_{i,a}$ extends to a section in \mathcal{H}_k .

By our discussion in Section 2, we know $\bar{\partial}\chi_i$ is supported in the region where $|\tau_a z_i^a e_i^{\otimes k}|_h$ is $\varepsilon(k)$ for all $a \leq k^{3/4}$, so by the above lemma $\|s_{i,a}\|^2 = 1 + \varepsilon(k)$. Similarly for $1 \leq a, b \leq k^{3/4}$ we have

$$\langle s_{i,a}, s_{j,b} \rangle = \delta_{ab}\delta_{ij} + \varepsilon(k).$$

We may assume $\{s_{i,a} | i = 1, \dots, d; a \leq k^{3/4}\}$ is orthonormal by possibly applying a linear transformation of the form $I + \varepsilon(k)$ which will only introduce an error of size $\varepsilon(k)$. Now we let $\{s_{\bar{j}}\}$ be an arbitrary orthonormal basis of the orthogonal complement of $\{s_{i,a} | i = 1, \dots, d; a \leq k^{3/4}\}$ in \mathcal{H}_k .

For each i , we denote by V_i the subset of U_i consisting of points such that $(\log \frac{1}{|z_i|^2})^{-1} \leq k^{-1/2} \log k$, and by W_i the subset of U_i consisting of points such that $(\log \frac{1}{|z_i|^2})^{-1} \leq k^{-3/8}$. As in Section 2, points in V_i has injectivity radius smaller than $\pi k^{-1/2} \log k$. For a point x outside $\bigcup_{i=1}^d V_i$, the usual proof of the Bergman kernel asymptotics goes through, and yield a uniform expansion (in the C^2 sense) that

$$\rho_k(x) = \frac{1}{2\pi} (k + \frac{S}{2} + O(k^{-1})),$$

and so

$$\omega_k = \rho_k \omega (1 - \frac{S}{2} k^{-1} + O(k^{-2})).$$

The following lemma essentially shows the “locality” of Bergman kernel in a neighborhood of the hyperbolic cusp.

Lemma 3.2. *On each W_i we have*

$$(3.1) \quad \rho_k(x) = (1 + \varepsilon(k))\rho_{k,0}(x).$$

Proof. For $x \in W_i \setminus V_i$, these follow from the fact both ρ_k and $\rho_{k,0}$ admit the same expansion. This also implies that $|s_{\tilde{\gamma}}|^2 \rho_{k,0}^{-1} = \varepsilon(k)$ on ∂V_i .

Now we focus on V_i . Recall that

$$\rho_k = \sum |s_{\alpha}|^2$$

The sections with sub-indices of the form (i, a) determines $\rho_{k,0}$ up to an error of $\varepsilon(k)$ within V_i as we have seen in the calculations for the model case. We only need to show that the contribution of each $s_{\tilde{\gamma}}$ to $\rho_k(x)$ is $\varepsilon(k)$ within V_i .

Let $f_{\tilde{\gamma}}$ be the holomorphic function corresponding to the trivialization of $s_{\tilde{\gamma}}$ within U_i . We consider the Taylor series of $f_{\tilde{\gamma}} = \sum a_i z^i$. By our choice of these $s_{\tilde{\gamma}}$, we may assume that $a_i = 0$ for $i \leq k^{3/4}$. Now we compare $f_{\tilde{\gamma}}$ with the function $\tau_a z^a$ with $a = k^{1/2} \log k$. The quotient $\frac{f_{\tilde{\gamma}}}{\tau_a z^a}$ is holomorphic. So we can apply the maximal module principle. When $\log \frac{1}{|z|^2} = k^{1/2} / \log k$, we know that

$$\frac{\rho_{k,0}}{|\tau_a z^a e_i^k|^2} = O\left(\sum_{c \in \mathbb{Z}} e^{-\frac{kc^2}{a^2}}\right) = O(\log k)$$

where e_i denote the local frame for L . On the other hand we have shown that $\frac{|s_{\tilde{\gamma}}|^2}{\rho_{k,0}} = \varepsilon(k)$ at $\log \frac{1}{x} = k^{1/2} / \log k$. Therefore on the boundary of V_i , we have

$$\left| \frac{f_{\tilde{\gamma}}}{\tau_a z^a} \right|^2 = \varepsilon(k)$$

So the same estimata holds within V_i . But we always have $\rho_{k,0} > |\tau_a z^a|^2$. So the contribution of $s_{\tilde{\gamma}}$ to $\rho_k(x)$ is indeed $\varepsilon(k)$ within V_i . So the lemma is proved. \square

Lemma 3.3.

$$\int_{V_i} \frac{|s_{\tilde{\gamma}}|^2}{\rho_k} \omega_k = \varepsilon(k).$$

Proof. We use the same trick at we did in the proof of last lemma, namely we compare $s_{\tilde{\gamma}}$ with $\tau_a z^a$, with $a = k^{1/2} \log k$. By the previous lemma we have

$$\int_{V_i} \frac{|s_{\tilde{\gamma}}|^2}{\rho_k} \omega_k \leq (1 + \varepsilon(k)) \int_{V_i} \frac{|f_{\tilde{\gamma}}|^2}{|\tau_a z^a|^2} \omega_k \leq \varepsilon(k) \log k \int_{V_i} \omega_k.$$

Now we simply notice that the total volume of ω_k is $kl - d$. \square

Lemma 3.4.

$$\int_{V_i} |s_{\tilde{\gamma}}|^2 \omega = \varepsilon(k)$$

Proof. This follows from the remark after lemma 2.1. More precisely, because of rapid decay, for $a > k^{3/4}$, we have

$$\frac{\int_{V_i} x^a |e_i|^{2k} \omega}{\int_{\log \frac{1}{x} \geq k^{1/4}} x^a |e_i|^{2k} \omega} = \varepsilon(k)$$

Again, we look at the terms of different degrees of $f_{\tilde{\gamma}}$, which are orthogonal to each other, when considered as functions for $\log \frac{1}{x} \geq k^{1/4}$. So we conclude

$$\int_{V_i} |s_{\tilde{\gamma}}|^2 \omega = \varepsilon(k) \int_{\log \frac{1}{x} \geq k^{1/4}} |s_{\tilde{\gamma}}|^2 \omega = \varepsilon(k)$$

\square

Using these we can estimate for $\alpha = \tilde{\gamma}$,

$$\begin{aligned}\mu_X(\alpha, \alpha) &= \int_{X \setminus \bigcup_i V_i} |s_\alpha|^2 (1 - \frac{S}{2} k^{-1} + O(k^{-2})) \omega + \int_{\bigcup_i V_i} |s_\alpha|^2 \rho_k^{-1} \omega_k \\ &= 1 - \frac{S}{2} k^{-1} + O(k^{-2}).\end{aligned}$$

Similarly for $\alpha = (i, a)$, we have analogous results as Lemma 3.3 and 3.4, by referring to the model case, and we also have

$$\mu_X(\alpha, \alpha) = 1 - \frac{S}{2} k^{-1} + O(k^{-2}).$$

Now for $\alpha = (i, a)$ with $a \leq 2k^{1/2} \log k$, we have $|s_\alpha|^2 \rho_k^{-1} = \varepsilon(k)$ outside W_i .

Lemma 3.5.

$$\mu_X(\alpha, \alpha) = (1 + \varepsilon(k)) \int_{W_i} \frac{|\tau_{i,a} z_i^a|^2}{\rho_{k,0}} \omega_{k,0} + \varepsilon(k)$$

Proof. The main point is that we can replace ω_k by $\omega_{k,0}$ in the expression of $\mu_X(\alpha, \alpha)$, with possibly an error term of size $\varepsilon(k)$. This follows from an argument similar to the proof of Lemma 3.4, using the results of Section 2. \square

By the results of Section 2, we know this is equal to $\frac{1}{2} + \varepsilon(k)$ if $a = 1$, and equal to $1 + O(k^{-1}(\log k)^{60})$ if $a > 1$.

Similarly, for the off-diagonal term of μ_X , we have

$$(3.2) \quad \mu_X(\alpha, \beta) = \begin{cases} \varepsilon(k), & \alpha = (i, a), \beta = (j, b), \alpha \neq \beta \\ O(k^{-2}) & \text{otherwise} \end{cases}$$

Finally we have

$$(3.3) \quad \mu_D(\alpha, \alpha) = \begin{cases} 1 + \varepsilon(k), & \alpha = (i, 1) \\ \varepsilon(k) & \text{otherwise} \end{cases}$$

Putting all these together, we see that

$$\|\mu_X + \frac{1}{2} \mu_D - \tilde{c}_k\|_2^2 = O(k^2 \cdot k^{-4} + O(k^{-2}(\log k)^{120} \cdot k^{1/2} \log k) = O(k^{-3/2}(\log k)^{121}).$$

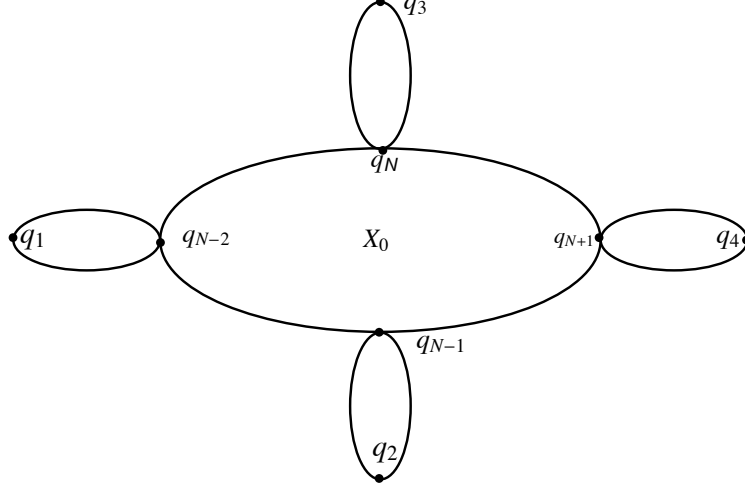
This finishes the proof of Theorem 1.2.

4. EXPLICIT STUDY OF CHOW STABILITY

We assume $X = \mathbb{P}^1$ and D the union of d distinct points p_1, \dots, p_d . Consider an embedding of (X, D) into \mathbb{P}^N using $H^0(X, L^k)$ where $L = [D]$ and $N = kd$.

Theorem 4.1. *Suppose $d \geq 2$, then (X, D) is λ -semistable if and only if $\lambda \in [\lambda_k, 1]$, and λ -stable if and only if $\lambda \in (\lambda_k, 1]$. Here $\lambda_k = 2/(d+1)$ when $k = 1$ and $\lambda_k = \frac{2kd+2}{3kd+d+1}$ when $k \geq 2$.*

Proof. Clearly (X, D) is always 1-balanced. A simple fact is that (c.f. [24]) the set of λ for which (X, D) is λ -semistable form an interval of the form $[\lambda_k, 1]$. The point is to determine λ_k . The pair (X, D) is not λ_k -balanced but there is another pair (X_0, D_0) in the closure of the $SL(N+1; \mathbb{C})$ orbit of (X, D) (in an appropriate Chow variety) which is λ_k -balanced. When $d = 2$ this is proved in [24] where X_0 is constructed as a chain of linear rational curves in \mathbb{P}^N . Now we focus on the case $d \geq 3$. We will construct these by induction. When $k = 1$, we let q_i be the i -th coordinate point of \mathbb{P}^d for $i = 1, \dots, d+1$. Then we let $D_0 = \{q_1, \dots, q_d\}$, and X_0 be the union of all lines connecting q_i with q_{d+1} . A straightforward calculation

FIGURE 1. A $\frac{2}{3}$ -balanced pair in \mathbb{P}^N with $D = \{p_1, p_2\}$ FIGURE 2. A λ_k -balanced pair in \mathbb{P}^N with $d = 4$ and $k \geq 2$

shows that (X_0, D_0) is λ_1 -balanced, and it is also easy to see that (X_0, D_0) is in the $SL(N+1; \mathbb{C})$ orbit of (X, D) . This shows that (X, D) is strictly λ_1 -semistable and hence we are done with $k = 1$. Now suppose the conclusion holds for $k = m-1$ and we consider the case $k = m$. Then again we denote by q_i the i -th coordinate point of \mathbb{P}^N for $i = 1, \dots, N+1$. Let X_0 be the union of a smooth rational normal curve Y in \mathbb{P}^{N-d} (viewed naturally as the a subspace of \mathbb{P}^N which contains q_1, \dots, q_{N-d+1}) which passes through the co-ordinate points, and the lines connecting the q_i with $q_{N-d+1+i}$. Let $D_0 = \{q_{N-d+1+j} | j = 1, \dots, d\}$, and $E = \{q_1, \dots, q_d\}$. This is in the closure of the $SL(N+1; \mathbb{C})$ orbit of (X, D) (this can be alternatively seen as a deformation to the normal cone). Now notice since $d \geq 3$ we have $\lambda_{m-1} < 2/3$, by the induction hypotheses we may assume (Y, E) is $2/3$ -balanced in \mathbb{P}^{N-d} . Then it is again a straightforward calculation that (X_0, D_0) is λ_m -balanced in \mathbb{P}^N . More precisely, we obtain this value of λ_m by solving the equation $\frac{1}{2}\lambda + (1-\lambda) = \lambda \frac{2N+d}{2(N+1)}$. By the same reason as the case $k = 1$ we see the conclusion holds for $k = m$. \square

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